**Chalker and Bernhardt (1992)**

So they aim to generalize the isotropy assumption that is the basis of the disorder average over the thin slice δL in the DMPK equation. First they introduce some numerical results that are indicative of the metal-insulator transition. They examine the behavior of the Lyapunov exponents νn, defined via λn ~ exp[2νnLz/ℓ] in the limit that Lz → ∞, with LT held fixed. He says that in 3D, as LT increases, the νn­ approach a limiting distribution. In the metallic state, we have: νn ~ 1/N, and so as L (and therefore LT = √N) increases to ∞, the density will be non-zero at the origin, and roughly flat. On the other hand, in the insulating state, the density of exponents will be non-zero at the origin and through till νmin, which determines the bulk localization length as L increases. Thus they observe that the transition is signalled by the opening up of a gap in the density of ν’s at the origin. At the critical point, the density does not change.

To establish a result later, he says that they assume a popular scaling ansatz for the conductance g = g(Lz/ξ), where physically speaking ξ would be the smallest localization length of the sample. ξ = ℓ/νmin ~ ℓ/(1/N) in the large N limit for metals and ~ ℓ/νmin, where νmin would be finite, in the large N limit for insulators. Anyway, in the metallic state, Ohm’s law prevails and g(x) ~ 1/x, whereas in the insulating state we have g(x) ~ e-2x. So this is presumptive evidence that the scaling law holds everywhere. Going back to metallic state, we may say that g ~ ξ/Lz ~ Aξ/Lz (filling in the A gratuitously sort of, but know it should be there). ξ/Lz. So we can write σ ~ ξ. This doesn’t make a whole lot of sense to me. I have to think about how the whole localization thing comes about.

**Real space scattering model**

With this in mind the idea is to come up with a model of the thin slice that predicts these properties of the exponents. To facilitate things we introduce an idealization of the sample; we model it has N ~ L2 single channels of length L. Thus scattering takes place in position space, rather than momentum space, as in reality a ‘channel’ is defined as a particular transverse momentum eigenvalue, not a longitudinal channel. Our model of scattering is this. Reflection is 180 degrees. There is no backscattering into other channels. On the other hand transmission into adjacent channels is possible. The likelihood of the transverse transmission is governed by a parameter f. They find numerically that by increasing the value of f one can induce a metal-insulator transition into the model – and even more importantly find that d = 2 is the lower critical dimension in this model (they assume TRS). In this model, they have derived an expression for **M**′ – the thin slice transfer matrix – out to order δL. It is,



I guess I can accept the idea of the matrix being expanded out in powers of L, since in Tartar’s paper, he has a stochastic differential equation for M, which if solved, would yield an expression like M(L), and the parameters in the formula would be random matrices themselves. So perhaps if we use Tartar’s formalism with the expectations in Chalker’s paper, such an expression for M΄ above would result. And further I could accept x and y being expandable in powers of L.

**Symmetry conditions satisfied?**

Does this matrix satisfy the symmetry requirements? We must have:



If x is Hermitian, y is symmetric, and x†y is symmetric, then this checks out. TRS requires M11\* = M22 and M12\* = M21. And this is automatically satisfied. Now all together, M has 2N2 + N d.o.f. And if these x, y requirements hold then x has N2 d.o.f., and y has N2 + N. So the # of d.o.f. match.

**x and y in terms of polar representation?**

Let’s consider the transfer matrix in the αβ representation. I’ll assume that we can expand α and β in powers of √δL. Then we’ll have:



And his expression is:



Comparison of ½ order terms leads us to identify:



Then for the first order terms to match we would need α1 = α1/22, and β1\* = β1/2\*α1/2. Who knows if these relationships hold, but there would have to be some relationship between different order coefficients since α and β can only have N2 and N2 + N d.o.f. respectively and so each successive coefficient in their expansion cannot be independent. And a relationship between the α and β coefficients shouldn’t be too surprising as there is one between u and υ since u­0υ0 = 1.

Let’s expand consider a related parameterization of M. Suppose we can write: α = eihL, and β = η√L. Then we have:



And comparing to Chalker:



we would say that x = h, and y = -iη. But again there is disagreement in the off-diagonals in the second term. Or compare to other Mello version:



Again, disagreement in the 2nd term off-diagonals. But anyway, this is sort of how it would work out.

**Let’s do it again**

So,



and expanding the first column,



So,



compared to:



So



obviously. And what we want is for u1/2υ0 + u0υ1/2 to equal 0. And this would require x = 0. hmmmm.

**Now let’s compare to Tartar**

Let’s take a brief look at Tartar:



If I do a Taylor series expansion of these expression, then I’d get, well, an analytic expansion? Not quite since, heuristically, U is on the order of 1/√L. So,



Then we can replace replace the dWi,i+1/2 with (1/2)dWi,i+1.



This wouldn’t seem to be consonant with their model, unless we eliminate the fast mode term. Perhaps we should do that. Yeah. Then we have, to first order:



Let’s consider the draft model. We have:



**Association of x and y with t and r**

He says that to first order in √δL, x and y are the forward transmission (well I think he means t – 1) and backwards reflection matrices, and apart from the –i factor, this would be born out by the hypothetical expansion of t and r΄ in the polar representation as



whereas:



So then his second order term would be (where cross terms are c.c.)

**Disorder averages of x and y**

So now he proposes:



b/c we assume that the phases are random. Furthermore,



In his simplified model he takes fab = f if a and b are nearest neighbors, but 0 otherwise. Thus f is basically the disorder parameter which controls the scaling. When f = 0, then we just have transmission down N 1D channels, which we already know is an insulator. When f gets large enough then coupling dominates and the sample becomes a metal when d > dc.

**Important correlations**

OK so now we need an expression for δλn­ out to order δL. Well,



and so:



Therefore the λ’s are simply the eigenvalues of M12M12T. So let’s form this expression and extract the eigenvalues. So



and so



and then,



I’ll call the perturbation V:



Then to extract the eigenvalues we use perturbation theory:



which we can write as:



So let’s form this matrix W:



Now writing out Wmn …



Now let’s consider <Wmn>Mʹ. This is:



and so in particular,



And now let’s consider <WmnWmʹnʹ>Mʹ,.



and so, in particular:



and,



**Expression for νn**

OK, so with our model of **M**′ we move to calculate the scaling of the expectation of the Lyapunov exponents. We have that in the asymptotic limit:



where at this point νn is a random matrix, just like λn is. So this is a stochastic expression. So then,



Solving for νn and expanding out to highest non-vanishing order (after expectations taken),



and so we get:



Now from above we have:



Let’s use the identity:



to simplify a little bit:



and this agrees with our previous results. And finally we need (δλn)2. Only the √δL/ℓ part of W will be the appropriate order here. So:



So we have:



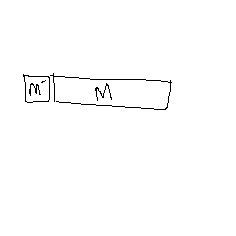
So it appears we have:



And this is what I’ve gotten before. But interesting that it appears here in this model. If we make the λm+1 >> λm argument, then this simplifies to:



Well I did this backwards and my transfer matrix is adding slices from right to left. I think that’s ok.



Now we can estimate the required behavior of the eigenvectors υαβ across all regimes through our knowledge of the how the νn’s act. For instance in the metallic regime we have |υab|2 ~ 1/N → Kmn = 1/N and so



But in the extreme insulating regime we must have |υαβ|2 is all 0 except at one index where it is 1. This would make Kmn ~ 0 for the most part, and Knn would be 1. So filling this in we have:



And so in the insulating regime, the νn are much larger basically.

**FP equation**

The only thing we need at this point is:



And if we make a RPA, then we’d have:



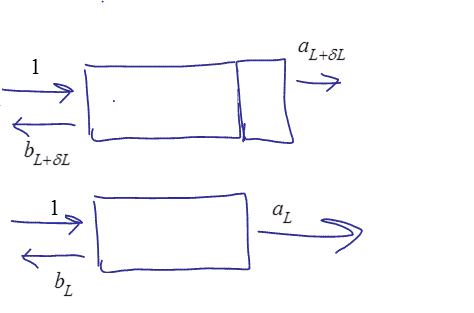
So the FP equation would be:



which is the usual Generalized FP equation.

**Recursive equation for total transmission coefficient**

Next he wants to develop an equation for <|υaN|4> in the metallic phase, where N is the greatest transmitting channel. To start we’ll write out a recursive equation for the transmission amplitude, aL. So we have:



which mathematically looks like:



This is probably most easily worked out in the polar representation. In polar form, the latter looks like:



and so we have:



where |1> is a column full of ones. To get tL+δL we’ll take a look at:



and so we have:



Under TRS, since t΄T = t, then we have, taking the transpose of both sides:



and so finally,



So this is a nice equation. On the other hand, I’m not sure it is the most convenient for evaluating averages since it is not in terms of x and y, but rather tδL and rδL. So let’s work it out in terms of x and y.



Expanding we’d have:



The perturbative expansion of an inverse matrix is …



So,



and so,



At this point we want to develop a recursion relation for the intensity = |a(L)|2. So we have:



So working this out, keeping terms only out to δL, which will survive the slice average, we have:



Now performing the average over the slice,



And so we have:



Now performing the average over the L-sized piece, he simplifies by decoupling the r΄ from the a(L), to get:



where Pab is the aforementioned average. This expression differs from his in the f-dependent term; he has something a little more complicated. He gets



Not sure how he got it honestly. He says that Fourier transform of Fαβ is given by:



for small q. Note **r**αβ is the displacement vector from **r**α to **r**β (each on same side of sample of course). Clearly it is independent of **r**α, the starting point, due to translational symmetry. Skipping ahead for now, he says that he has modelled Pab (theoretically?) via a classical diffusion process, and obtains the result for its Fourier transform:

